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Sigma Models and Minimal Surfaces

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Abstract. Correspondence is established between sigma models, minimal surfaces and the Monge–Ampére equation. The Lax pairs of the minimality condition of the minimal surfaces and the Monge–Ampére equations are given. Existence of infinitely many nonlocal conservation laws is shown and some Bäcklund transformations are also given.

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1. In a recent paper [1], we investigated the classical integrability of the sigma models in a non-Riemannian background and gave their one-soliton Bäcklund transformations. In particular, two-dimensional sigma models with a Wess–Zumino term have been studied in detail.

Let M be a two-dimensional manifold with local coordinates $x^{\mu} = (t, x)$ and $\Lambda^{\mu\nu}$ be the components of a tensor field in M. Let P be a 2×2 matrix with $\det(P) = 1$. We assume that P is a Hermitian ($P^{\dagger} = P$) matrix. Then the sigma model we consider is given as

$$\frac{\partial}{\partial x^{\alpha}} \left(\Lambda^{\alpha\beta} P^{-1} \frac{\partial P}{\partial x^{\beta}} \right) = 0. \tag{1}$$

The integrability of the above equation has been studied in [1]. The uniqueness of the solutions of these equations under certain boundary conditions is given in [2]. In these works, the matrix function P and the tensor $\Lambda^{\alpha\beta}$ were considered to be independent. We have classified possible forms of the tensor $\Lambda^{\alpha\beta}$ under the condition of integrability.

In some cases, these two quantities may be related. Such a relation may provide some interesting equations. In this Letter, we are interested in the integrability property of such cases. As an example, let $P = g$, where g is matrix representing the metric $g_{\alpha\beta}$ symmetric with respect to the lower indices. Also letting $\Lambda^{\alpha\beta} = g^{\alpha\beta}$, the inverse components of the metric $g_{\alpha\beta}$, then (1) becomes

$$\frac{\partial}{\partial x^{\alpha}} \left(g^{\alpha\beta} g^{-1} \frac{\partial g}{\partial x^{\beta}} \right) = 0. \tag{2}$$

In the theory of surfaces in $\mathbb{R}^3$ there is a class, the minimal surfaces of which have special importance both in physics and mathematics [3, 4]. Let $S = \{(t, x, z) \in \mathbb{R}^3; z = h(t, x)\}$ define a surface $S \in \mathbb{R}^3$ which is the graph of a differentiable function $h(t, x)$. This surface is called minimal if h satisfies the condition

$$(1 + h,_x^2)\, h,_{tt} - 2h,_x\, h,_t\, h,_{xt} + (1 + h,_t^2)\, h,_{xx} = 0, \tag{3}$$

The Gaussian curvature K of the surface S is given by

$$K = \frac{h,_{xx}\, h,_{tt} - h,_{xt}^2}{(1 + h,_x^2 + h,_t^2)^2}. \tag{4}$$

2. The sigma model equation (1) is integrable for certain choices of the tensor field $\Lambda^{\alpha\beta}$. In two dimensions, the integrability conditions on this tensor are given by

$$\partial_{\alpha} \left(\frac{1}{\sigma}\, \Lambda^{\alpha\beta}\, \partial_{\beta}\, \sigma \right) = 0, \qquad \partial_{\alpha} \left(\frac{1}{\sigma}\, \Lambda^{\beta\alpha}\, \partial_{\beta}\, \phi \right) = 0, \tag{5}$$

where σ is the determinant and ϕ is its antisymmetric part of the tensor field $\Lambda^{\alpha\beta}$. Hence, by letting $\Lambda^{\alpha\beta} = g^{\alpha\beta}$, the above conditions are trivially satisfied because $\sigma = 1$ and $\phi = 0$. Then using the approach developed in [1], it is straightforward to show that (2) is also integrable. This leads to the following proposition:

PROPOSITION 1. *The Lax pair of (2) is*

$$\varepsilon^{\alpha\beta}\, \frac{\partial}{\partial x^{\beta}}\, \Psi = \frac{1}{k^2 + 1}\, (k\, g^{\alpha\beta} - \varepsilon^{\alpha\beta})\, g^{-1}\, \frac{\partial g}{\partial x^{\beta}}\, \Psi, \tag{6}$$

provided $\det(g) = 1$ *and* $g_{\alpha\beta}$ *is symmetric. Here* k *is an arbitrary constant (the spectral parameter) and* $\varepsilon^{\alpha\beta}$ *is the Levi-Civita tensor with* $\varepsilon^{12} = 1$.

A standard parametrization of $g_{\alpha\beta}$ may be given as

$$\mathrm{d}s^2 = g_{\alpha\beta}\, \mathrm{d}x^{\alpha}\, \mathrm{d}x^{\beta} = \frac{1}{w}\, [(1 + a^2)\, \mathrm{d}t^2 + 2\, a\, b\, \mathrm{d}x\, \mathrm{d}t + (1 + b^2)\, \mathrm{d}x^2], \tag{7}$$

where $x^{\alpha} = (t, x)$, a and b are differentiable functions of t and x and

$$w^2 = 1 + a^2 + b^2. \tag{8}$$

PROPOSITION 2. *Let h be a differentiable function of t and x and let $a = h_{,t}$ and $b = h_{,x}$, then the minimality condition (3) solves the sigma model equation (2).*

This result is interesting and also very important. We shall give the Lax pair (6) in a more detailed way, but before that we write the minimality condition in a covariant way. The metric on this minimal two-dimensional surface S is

$$(\mathrm{d}s)^2_m = g_{(m)\,\mu\,\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{9}$$

$$= (1 + h_{,t}^{\,2})\,\mathrm{d}t^2 + 2\,h_{,t}\;h_{,x}\;\mathrm{d}x\,\mathrm{d}t + (1 + h_{,x}^{\,2})\,\mathrm{d}x^2. \tag{10}$$

Then the minimality condition (3) may be written covariantly as

$$g^{\alpha\,\beta}_{(m)}\,\partial_\alpha\,\partial_\beta\,h = 0. \tag{11}$$

Since $g_{(m)\;\mu\nu} = \delta_{\mu\nu} + h_{,\mu}\;h_{,\nu}$, where $\delta_{\mu\nu}$ is the Kronecker delta symbol, (11) is also equivalent to

$$\partial_\alpha\,(\sqrt{g_{(m)}}\,g^{\alpha\,\beta}_{(m)}) = 0, \tag{12}$$

where $g_{(m)}$ is the determinant of the metric $g_{(m)\,\alpha\,\beta}$ on S. S is embedded in a flat three-dimensional Euclidean space $\mathbb{R}^3$ with metric $\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}z^2$. The minimality conditions (11) and(12) are equivalent to the harmonicity of the function $h(t, x)$ with respect to the metric of S

$$\partial_\alpha\,(\sqrt{g_{(m)}}\,g^{\alpha\,\beta}_{(m)}\,\partial_\beta\,h) = 0. \tag{13}$$

In the language of harmonic mappings of Riemannian manifolds [5], Equations (11), (12), and (13) imply that the mapping $x^\alpha\colon S \to S$ is harmonic. Here we would like remark that the nonlinear partial differential equation (3) describing the minimality condition of a two-dimensional surface S is a special case of the sigma model equation (2). Hence, it straightforward to conclude that Equation (3) is integrable and its Lax pair is given in (6). We shall now give this Lax pair more explicitly. Let $A = g^{-1}\,\partial_t\,g$ and $B = g^{-1}\,\partial_x\,g$ be two 2×2 matrices with components

$$A^1_1 = \frac{1}{w^2}\,[p(1 + q^2)r - q(1 + p^2)s], \tag{14}$$

$$A^1_2 = \frac{1}{w^2}\,[q(1 + q^2)r + p(1 - q^2)s], \tag{15}$$

$$A^2_1 = \frac{1}{w^2}\,[q(1 - p^2)r + p(1 + p^2)s], \tag{16}$$

$$A^2_2 = -\frac{1}{w^2}\,[p(1 + q^2)r - q(1 + p^2)s], \tag{17}$$

$$B_1^1 \;=\; \frac{1}{w^2}\,[p(1+q^2)s - q(1+p^2)t], \tag{18}$$

$$B_2^1 \;=\; \frac{1}{w^2}\,[q(1+q^2)s + p(1-q^2)t], \tag{19}$$

$$B_1^2 \;=\; \frac{1}{w^2}\,[q(1-p^2)s + p(1+p^2)t], \tag{20}$$

$$B_2^2 \;=\; -\frac{1}{w^2}\,[p(1+q^2)s - q(1+p^2)t], \tag{21}$$

where we have used the same notation as used in [4]

$$p = h_t, \qquad q = h_x, \qquad r = h_{tt}, \qquad s = h_{tx}, \qquad t = h_{xx}, \tag{22}$$

$$w^2 = 1 + p^2 + q^2. \tag{23}$$

Then the Lax pair becomes

$$\Psi_{,x} = -\frac{1}{k^2+1}\,[k(-r'\,A + q'\,B) + B]\,\Psi, \tag{24}$$

$$\Psi_{,t} = -\frac{1}{k^2+1}\,[k(-q'\,A + p'\,B) + A]\,\Psi, \tag{25}$$

where k is the spectral parameter p', q' and r' are given by

$$p' = \frac{1+p^2}{w}, \qquad q' = \frac{p\,q}{w}, \qquad r' = \frac{1+q^2}{w}. \tag{26}$$

Integrability of Equations (24) and (25) gives

$$(r'\,A - q'\,B)_{,t} + (p'\,B - q'\,A)_{,x} = 0, \tag{27}$$

$$A_{,x} - B_{,t} = [A, B]. \tag{28}$$

The first of the above equations is identical with the minimality condition (3) and the second one is a trivial identity.

3. From the Lie symmetries of the minimality condition, it may be possible to find some conservation laws. Some of these are given by [4]

$$\left(\frac{q}{w}\right)_{,x} + \left(\frac{p}{w}\right)_{,t} = 0, \tag{29}$$

$$\left(\frac{p\,q}{w}\right)_{,x} + \left(-\frac{(1+q^2)}{w}\right)_{,t} = 0, \tag{30}$$

$$\left(\frac{(1+p^2)}{w}\right)_{,x} + \left(-\frac{p\,q}{w}\right)_{,t} = 0. \tag{31}$$

These conservation laws are local in the following sense. In general, any conservation law can be written as $X_{,x} = T_{,t}$, where X and T are functions of h, p, q, r, s, t, and higher derivatives of these functions with respect x and t. Such conservation laws are the local ones. In the case of nonlocal conservation laws, the functions X and T depend, in addition to h, p, q, r, s, t, and higher derivatives of these functions with respect x and t, upon the integrals of these variables with respect to x and t. One can find such conservation laws in this case as well. Let us assume that the function Ψ in (24)–(25) is analytic in the parameter k and can be expanded as

$$\Psi = \Psi_0 + k\,\Psi_1 + k^2\,\Psi_2 + \cdots. \tag{32}$$

Then Equations (24)–(25) imply

$$\Psi_0 = g^{-1}, \tag{33}$$

$$(g\,\Psi_1)_{,x} = -g\,M\,g^{-1}, \tag{34}$$

$$(g\,\Psi_1)_{,t} = -g\,N\,g^{-1}, \tag{35}$$

$$(g\,\Psi_2)_{,x} = g_x\,g^{-1} - g\,M\,g^{-1}\,D_x^{-1}\,g\,M\,g^{-1}, \tag{36}$$

$$(g\,\Psi_2)_{,t} = g_t\,g^{-1} - g\,N\,g^{-1}\,D_x^{-1}\,g\,N\,g^{-1}, \tag{37}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

where D_x^{-1} and D_t^{-1} are, respectively, the inverse operators of the total derivatives with respect to x and t and

$$M = -r'\,g^{-1}g_{,t} + q'\,g^{-1}g_{,x}, \qquad N = -q'\,g^{-1}g_t + p'\,g^{-1}g_x. \tag{38}$$

Hence, we have now infintely many conservation laws with functions X_n and T_n for all $n = 0, 1, 2, \ldots$. The first two members may be given from the above equations:

$$X_0 = M, \qquad T_0 = N, \tag{39}$$

$$X_1 = g^{-1}\,g_{,x} + (D_x^{-1}\,M)\,M, \qquad T_1 = g^{-1}\,g_{,t} + (D_t^{-1}\,N)\,N, \tag{40}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

In this way, one can find infinitely many nonlocal conservation laws.

4. The Bäcklund transformation obtainable from the Lax pair (24)–(25) is not suitable because the correspondence between the new and old solutions will be of the same degree as of that of the minimality condition. Hence, one has to solve a second-order differential equation which is as hard as the original equation. Instead, we shall mention two interesting nonauto-Bäcklund transformations.

The solution of (3) can be expressed in terms of two harmonic functions.

PROPOSITION 3. *Let x and t be harmonic functions of u and v and let a differentiable function $h(t, x)$ be defined by*

$$[1 + p^2]\, t,_u = -w\, x,_v - q\, p\, x,_u\,, \qquad [1 + p^2]\, t,_v = -w\, x,_u - q\, p\, x,_v$$

Then the function $h(t, x)$ is a harmonic function of u and v if and only if it satisfies the minimality condition (3).

This proposition implies that the function $h(t, x)$ can be constructed in terms of two harmonic functions $t(u, v)$ and $x(u, v)$. The function $h(t, x)$ obtained in this way automatically satisfies the minimality condition (3). In this case, the metric (10) on the two-dimensional surface S takes the conformally flat form

$$\mathrm{d}s^2_{(m)} = w^2 \left(\frac{x,_u^2 + x,_v^2}{1 + p^2} \right) (\mathrm{d}u^2 + \mathrm{d}v^2). \tag{41}$$

Here we understand that the minimality condition (3) arises from a sigma model so that the target and base space metrics are the same. Such a sigma model has a Lax pair defined in the linear equation (6) in Proposition 1 (or in (24) – (25)). This Lax pair may be used to construct Bäcklund transformation for Equation (3) (the minimality condition). Instead of following such a direction, we find the Bäcklund transformation by defining a new 2×2 matrix function Q,

$$g^{\alpha\beta}\, g^{-1}\, \partial_\beta\, g = \varepsilon^{\alpha\beta}\, \partial_\beta\, Q \tag{42}$$

PROPOSITION 4. (a) *Equation corresponding to the matrix Q is*

$$\partial_\alpha\, (g^{\alpha\beta}\, \partial_\beta\, Q) - \varepsilon^{\alpha\beta}\, \partial_\alpha\, Q\; \partial_\beta\, Q = 0. \tag{43}$$

(b) *The corresponding linear equation is*

$$\varepsilon^{\alpha\beta}\, \partial_\beta\; \Psi = \frac{1}{k^2 + 1}\, (k\, \varepsilon^{\alpha\beta} + g^{\alpha\beta})\, \partial_\beta Q\;\, \Psi \tag{44}$$

There is a second Bäcklund transformation for Equation (3) obtainable simply by using either (6) or (44).

PROPOSITION 5. *Let $z = h(t, x)$ define a minimal surface embedded in the three dimensional Euclidean space $\mathbb{R}^3$. The following transformation:*

$$\frac{h_{,x}}{w} = \psi_{,t}\,, \qquad \frac{h_{,t}}{w} = -\psi_{,x}\,, \tag{45}$$

maps the minimality condition (3) *to the equation*

$$(1 - \psi_{,x}^2)\,\psi_{,tt} + 2\psi_{,x}\;\psi_{,t}\;\psi_{,xt} + (1 - \psi_{,t}^2)\,\psi_{,xx} = 0. \tag{46}$$

This equation defines a minimal surface $S' = ((t, x, w') : w' = \psi(t, x))$. S' is embedded in a three-dimensional Minkowski space M_3 with the metric $\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{d}x^2 - \mathrm{d}\,{w'}^2$. The metric on S' is given by

$$\begin{aligned} \mathrm{d}s'^2_{(m)} &= g'_{(m)\,\alpha\,\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta \\ &= (1 - \psi_{,t}^2)\,\mathrm{d}\,t^2 - 2\,\psi_{,t}\;\psi_{,x}\;\mathrm{d}\,x\,\mathrm{d}\,t + (1 - \psi_{,x}^2)\,\mathrm{d}\,x^2. \end{aligned} \tag{47}$$

The minimality condition (46) for the surface S' may be written as

$$g'^{\alpha\,\beta}_{(m)}\;\psi_{,\alpha\,\beta} = 0. \tag{48}$$

As an illustration to the above transformation (45), we can give the following nontrivial examples. The minimal surfaces

$$\psi = \frac{1}{\lambda}\,[\ln\,\cosh(\lambda\,t) - \ln\,\cosh(\lambda\,x)]$$

and

$$h = \frac{1}{\lambda}\,\cos^{-1}[\sinh(\lambda\,t)\;\sinh(\lambda\,x)]$$

are transformable to each other. Here λ is a nonvanishing constant.

Finally we would like to mention another Bäcklund transformation which maps solutions of the minimality condition to the solutions of the Monge–Ampere equation. This is given by the following proposition:

PROPOSITION 6. *Let the function $h(t, x)$, with enough differentiability, satisfy the minimality condition (3), then the metric $g_{\mu\,\nu} = (1/w)\,g_{(m)\,\mu\,\nu}$ satisfies the condition*

$$\partial_\alpha\,g_{\mu\,\nu} = \partial_\nu\,g_{\mu\,\alpha}, \tag{49}$$

which also implies that

$$g_{\mu\,\nu} = \partial_\mu\,\partial_\nu\,u, \tag{50}$$

where $u(t, x)$ is enough differentiable function of t, x satisfying the equation

$$\mathrm{Det}\,(\partial_\mu\,\partial_\nu\,u) = u_{,tt}\,u_{,xx} - u_{tx}^2 = 1. \tag{51}$$

This is the equation known as the Monge–Ampére equation which is also integrable and its Lax Pair can be easily obtained by using (50) in (6) or in (24)–(25). Hyperbolic minimal surfaces also have similar correspondence with the Monge–Ampére equation. Using (46) and (47) we have

$$g'_{\mu\,\nu} = \partial_\mu\,\partial_\nu\,u, \tag{52}$$

with

$$\mathrm{Det}\,(\partial_\mu\,\partial_\nu\,u) = u_{,tt}\,u_{,xx} - u_{tx}^2 = 1. \tag{53}$$

which does not give the hyperbolic Monge–Ampére equation as expected. The correspondence between the minimal surfaces in $\mathbb{R}^3$ and the Monge–Ampére equation is mentioned in [6, 7]. The correspondence between the Born–Infeld and the hyperbolic Monge–Ampére equation is mentioned in [8].

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